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Analysis of a strongly sheared, nearly homogeneous turbulent shear flow

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Harris, Graham & Corrsin (1977) have measured the properties of the quasi-homogeneous turbulence field induced by a large mean shear, and in analysing their measurements they neglect the diagonal components of the mean-field contribution to the pressure–strain correlation. Their measurements are re-analysed using the recommendations of Gibson & Launder (1978) for modelling this correlation, the imbalance between production and dissipation being allowed for by the algebraic modelling technique of Rodi (1976): the experimental data give strong support to Rodi's basic assumption.

The data suggest that the only substantial defect in the Gibson–Launder model is its failure to predict the anisotropy measured in the 23 plane normal to the mean flow. The longitudinal predictions are good, and those for the shear (12) component are much improved when measured values of the anisotropy are substituted into the calculation. This analysis does not suggest any clear need for a nonlinear representation of the pressure–strain correlation. However, the most general linear representation of the mean-field term is even more complex than the analysis of Launder, Reece & Rodi (1975) would suggest: their model is disproved by an example.

Attempts to deduce the dissipation directly from the experiments, rather than by energy balance, are not very successful.

1. Introduction

The experiment of Champagne, Harris & Corrsin (1970; hereinafter called CHC) on the nearly homogeneous turbulent field associated with quasi-uniform mean shear is of central importance in turbulence modelling, since the difficult pressure–strain term can be inferred directly from it. Harris *et al.* (1977; hereinafter called HGC) have extended this type of measurement to much higher rates of shear ($48{\cdot}0\,\mathrm{s}^{-1}$ as compared to $12{\cdot}4\,\mathrm{s}^{-1}$ in the CHC experiments†). Their analysis suggests that there are serious defects in the standard model of Rotta (1951) for the pressure–strain term. This is surprising, since HGC show that their high-shear experiments are compatible with the low-shear experiments of CHC while Launder (Launder *et al.* 1975; Gibson & Launder 1978) finds that the earlier CHC experiment is, by and large, compatible with the Rotta model.

The aim of this paper is to re-analyse the high-shear experiments of HGC, using

† HGC have issued an errata sheet which modifies some of the experimental numbers quoted in their paper. The modified values are used in the present paper.

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Mean shear $\partial U_1/\partial x_2 = 48{\cdot}0\ \mathrm{s}^{-1}$	
$u_1' = \langle u_1^2\rangle^{\frac{1}{2}} = 64\ \mathrm{cm\ s}^{-1}$	$\lvert m_{12}\rvert = \langle -u_1u_2\rangle/k = 0{\cdot}297$
$u_2' = 40{\cdot}4\ \mathrm{cm\ s}^{-1}$; $u_3' = 49{\cdot}5\ \mathrm{cm\ s}^{-1}$	Centre-line velocity $\overline{U}_c = 1240\ \mathrm{cm\ s}^{-1}$
$k = \frac{1}{2}\langle u_iu_i\rangle = 4096\ \mathrm{cm^2\ s^{-2}}$	$U_c\, du_1'^2/dx_1 = 1{\cdot}96\times 10^4\ \mathrm{cm^2\ s^{-3}}$
Reynolds stress $\langle -u_1u_2\rangle = 1217\ \mathrm{cm^2\ s^{-2}}$	$\overline{U}_c du_2'^2/dx_1 = 0{\cdot}88\times 10^4\ \mathrm{cm^2\ s^{-3}}$
$m_{11} = u_1'^2/k - \frac{2}{3} = 0{\cdot}336$	$\overline{U}_c\, du_3'^2/dx_1 = 1{\cdot}32\times 10^4\ \mathrm{cm^2\ s^{-3}}$
$m_{22} = u_2'^2/k - \frac{2}{3} = -0{\cdot}268$	$\overline{U}_c\, dk/dx_1 = 2{\cdot}08\times 10^4\ \mathrm{cm^2\ s^{-3}}$
$m_{33} = u_3'^2/k - \frac{2}{3} = -0{\cdot}068$	$\overline{U}_c\, d\langle -u_1u_2\rangle/dx_1 = 0{\cdot}62\times 10^4\ \mathrm{cm^2\ s^{-3}}$

TABLE I. The basic data.

algebraic modelling (Rodi 1976) to account for the strong imbalance between production and dissipation: in particular, their over-rigorous interpretation of the Rotta model will be corrected.

2. The experimental data

This is taken from table 3 of HGC, as amended by the errata sheet (see table 1). The mean flow is in the x_1 direction and the mean velocity gradient in the x_2 direction. The mean and fluctuating velocities are denoted by U_i and u_i. The data is taken 'just at the downstream end of the test section' (presumably $x_1/h = 11{\cdot}0$, though this is not stated) where 'it appears that the turbulence has attained an asymptotic state'. It was considered impractical to analyse the evolution of the flow, interesting though this would be, since the flow properties are tabulated for this single value of x_1/h only.

An attempt is made in section (*c*) of the appendix to deduce the dissipation ϵ directly from the experimental data. The results are so inaccurate that one cannot infer from them a reasonable value of P/ϵ, the ratio of production to dissipation. ϵ must therefore be inferred from the energy equation

$$\text{advection } \overline{U}_c\frac{dk}{dx_1} = \text{production } P - \text{dissipation } \epsilon, \tag{1}$$

$$2{\cdot}08 \quad = \quad 5{\cdot}84 \quad - \quad 3{\cdot}76 \qquad (10^4\,\mathrm{cm^2\,s^{-3}});$$

the production being computed from

$$P = \langle -u_1u_2\rangle\frac{\partial U_1}{\partial x_2}. \tag{2}$$

(In the HGC paper, the dissipation is quoted as $3{\cdot}28\times 10^4\,\mathrm{cm^2\,s^{-3}}$; this value is compatible with equation (1), and is clearly inferred from it. When $\overline{U}_c$ is increased from $44{\cdot}0\,\mathrm{s}^{-1}$, as quoted in the paper, to $48{\cdot}0\,\mathrm{s}^{-1}$ as quoted in the errata sheet, the dissipation as calculated from equation (1) is increased to $3{\cdot}76\times 10^4\,\mathrm{cm^2\,s^{-3}}$: the value quoted in the errata sheet, $3{\cdot}35\times 10^4\,\mathrm{cm^2\,s^{-3}}$, appears to be in error.) Throughout the paper we shall take

$$\epsilon = 3{\cdot}76\times 10^4\,\mathrm{cm^2\,s^{-3}}; \quad P/\epsilon = 1{\cdot}55. \tag{3}$$

Strictly there is also a diffusion term in equation (1). Harris (see HGC) has measured the velocity contribution to this term, and finds it to be less than 3 % of the advection. The estimate of the whole term, made in section (*b*) of the appendix with the aid of a model, is even smaller. The diffusion term is therefore ignored.

3. Models for the pressure–strain term

Rotta (1951) showed that there were two contributions to the pressure–strain term

$$\phi_{ij} = \left\langle p\left(\frac{\partial u_i}{\partial x_j}+\frac{\partial u_j}{\partial x_i}\right)\right\rangle = \phi_{ij,1}+\phi_{ij,2}, \tag{4}$$

p being the fluctuating pressure. The first, $\phi_{ij,1}$, is a triple correlation of the fluctuating velocity while the second, $\phi_{ij,2}$, is a product of the mean velocity gradient with a pair correlation of the fluctuating velocity. Most later workers have used Rotta's model for the first term

$$\phi_{ij,1} = -c_1\frac{\epsilon}{k}(\langle u_i u_j\rangle - \tfrac{2}{3}k\delta_{ij}) \tag{5}$$

and Gibson & Launder recommend

$$c_1 = 1{\cdot}8.$$

We shall follow this recommendation.

However Lumley (1978, and see this paper for references to his earlier publications) has shown that c_1 cannot really be a constant. In principle at least, it must be a functional of invariants formed from the deviator of the stress tensor

$$m_{ij} = \frac{1}{k}\langle u_i u_j\rangle - \tfrac{2}{3}\delta_{ij}.$$

Chung & Adrian (1979) have based a turbulence model on these principles, but it does not seem to perform any better than the linear model of Launder *et al.* (1975), whose recommendations are followed in this paper with satisfactory results.

The mean field term $\phi_{ij,2}$ does not involve a closure problem and is amenable to direct calculation: nonetheless, there is much less agreement as to how this 'easier' term should be modelled. Following Crow (1968) and Naot, Shavit & Wolfshtein (1970), Gibson & Launder recommend

$$\phi_{ij,2} = -\,c_2(P_{ij}-\tfrac{2}{3}\delta_{ij}P), \tag{6}$$

where

$$P_{ij} = -\langle u_i u_k\rangle\frac{\partial U_j}{\partial x_k}-\langle u_j u_k\rangle\frac{\partial U_i}{\partial x_k} \tag{7}$$

and

$$P = \tfrac{1}{2}P_{ii}. \tag{8}$$

When the turbulence is homogeneous and isotropic, the model (6) is exact with

$$c_2 = 0{\cdot}6.$$

Following Launder *et al.* (1975), Gibson & Launder recommend this value for general use, and we adopt this recommendation. In the simple geometry of the CHC and HGC experiments, the model (6) gives

$$\left.\begin{aligned}\phi_{11,2} &= -\tfrac{4}{3}c_2 P,\\ \phi_{22,2} &= \phi_{33,2} = +\tfrac{2}{3}c_2 P,\\ \phi_{12,2} &= +c_2\langle u_2^2\rangle\frac{dU_1}{dx_2},\end{aligned}\right\} \tag{9}$$

where P is, as before, the rate of production of turbulent energy (see equation (2)).

HGC say that Rotta's hypothesis makes the diagonal components of $\phi_{ij,2}$ zero, and analyse their data accordingly. It is indeed true that if the turbulence were strictly isotropic the model (6) would make the diagonal terms zero, because there could be no Reynolds stress and no production. But this is not what is assumed. Gibson & Launder formulate the model and calculate the constant c_2 as though the turbulence were isotropic, and then apply the result quite generally.

This is analogous to the use of a quasi-normal procedure to close the turbulence equations. If the turbulence field were truly normal, there could be no inertial transfer by the triple terms, but the assumption does not go so far: it merely states that the quadruple and pair correlations are related as they would be if the turbulence were normal.

Equations (5) and (6) make ϕ_{22} equal to ϕ_{33}, and this implies $\langle u_2^2 \rangle = \langle u_3^2 \rangle$. The experiments show that there is substantial anisotropy in the 23 plane normal to the mean flow, and Launder *et al.* have devised a more elaborate model for $\phi_{ij,2}$ in an attempt to remove this discrepancy. They write

$$\phi_{ij,2} = (a_{lj}^{mi} + a_{li}^{mj}) \frac{\partial U_l}{\partial x_m}, \tag{10}$$

which is correct if the flow is homogeneous, and they assume that the a_{li}^{mj} are general linear functionals of the $\langle u_r u_s \rangle$. There are five possible types of term and four kinetic constraints on them, so that the model contains one arbitrary parameter γ_2.† For the simple flow situation of the CHC/HGC experiments, the Launder *et al.* model reduces to

$$\left.\begin{aligned} \phi_{11,2} &= -\tfrac{4}{3}P\frac{8-3\gamma_2}{11}, \\ \phi_{22,2} &= +\tfrac{2}{3}P\frac{12-15\gamma_2}{11}, \\ \phi_{33,2} &= +\tfrac{2}{3}P\frac{9\gamma_2+6}{11}. \end{aligned}\right\} \tag{11}$$

$\phi_{33,2}$ is no longer equal to $\phi_{22,2}$ and, with two independent parameters, the model can be adjusted to give any desired degree of anisotropy. Launder *et al.* choose these parameters to produce agreement with the CHC experiments, and Gibson (private communication) has shown that this parameter set does not reproduce the anisotropy measured in the HGC experiment. (The Gibson & Launder model contains only one anisotropy parameter, $1-c_2/c_1$.)

It is shown in the appendix that assumption of Launder *et al.* for the coefficients a_{lj}^{mi} is not general enough. Since their model is fairly complex and does not have a secure theoretical basis, the model of Gibson & Launder will be used to analyse the HGC experiments.

4. Energy imbalance and algebraic modelling

Gibson & Launder assume that the flow is in local equilibrium, and that all transport (advective + diffusive) terms in the equations of motion are zero. In general, a flow with constant mean shear is not of this restricted type, and the turbulence level

† Launder *et al.* call this parameter c_2. In the present paper the symbol c_2 is reserved for the quantity defined by equation (6), in conformity with the notation of Gibson & Launder.

will increase or decrease as the field is advected downstream. Therefore it will generally be necessary to include the advection terms in the equation of motion: section (*b*) of the appendix shows that diffusion is negligible.

Fortuitously the CHC experiment is very near to local equilibrium, but the data in §2 shows that this is far from true in the HGC experiment, where, from equation (3),

$$P = 1{\cdot}55\epsilon.$$

This imbalance is too large to ignore: fortunately it can be allowed for, without making the analysis much more complicated, by using the algebraic modelling technique of Rodi (1976). (This procedure is advocated by, for instance, Meroney (1976).)

The balance equation for $\langle u_i u_j \rangle$ may be written as

$$\underset{\text{transport}}{T_{ij}} = \underset{\text{production}}{P_{ij}} - \underset{\text{dissipation}}{\epsilon_{ij}} + \underset{\text{pressure-strain}}{\phi_{ij}} \tag{12}$$

and in particular, since $\phi_{ii} = 0$, the transport of the turbulent energy $k = \frac{1}{2}\langle u_i u_i \rangle$ is given by

$$T_k = P - \epsilon, \tag{13}$$

where

$$P = \tfrac{1}{2}P_{ii}, \quad \epsilon = \tfrac{1}{2}\epsilon_{ii}$$

(cf. equation (8)). The algebraic modelling assumption is that

$$T_{ij} = \frac{\langle u_i u_j \rangle}{k} T_k = \frac{\langle u_i u_j \rangle}{k}(P - \epsilon) \tag{14}$$

and it is so called because it eliminates the differentials implicit in T_{ij}, leaving a set of algebraic relations; examples are given in the next section.

Table 2 compares the actual measurements of

$$\overline{U}_c \frac{d}{dx_1}\langle u_i u_j \rangle$$

with values of

$$\frac{\langle u_i u_j \rangle}{k} T_k = \frac{\langle u_i u_j \rangle}{k} \overline{U}_c \frac{dk}{dx_1}$$

computed from the measured data. The agreement is impressive and it is clear that Rodi's assumption (14) is a good approximation, at least in the simple geometry of the HGC experiment.

Component	$\overline{U}\dfrac{d}{dx_1}\langle u_i u_j \rangle$	$\dfrac{\langle u_i u_j \rangle}{k}\overline{U}_c \dfrac{dk}{dx_1}$
$\langle u_1^2 \rangle$	1·96 cm² s⁻³	2·09 cm² s⁻³
$\langle u_2^2 \rangle$	0·88	0·83
$\langle u_3^2 \rangle$	1·32	1·24
$\langle -u_1 u_2 \rangle$	0·62	0·62

TABLE 2. Test of algebraic modelling.

Quantity	Measured CHC	Predicted $P/\epsilon = 1$	Predicted $P/\epsilon = 1{\cdot}55$	Measured HGC
m_{11}	0·27	0·296	0·352	0·336
m_{22}	−0·18	−0·148	−0·176	−0·268
m_{33}	−0·09	−0·148	−0·176	−0·068
$\lvert m_{12}\rvert$	0·34	0·339	0·360	0·297

TABLE 3. Comparison of measured and predicted anisotropy parameters.

5. Prediction of the anisotropy

In the balance equation (12), we put

$$\epsilon_{ij} = \tfrac{2}{3}\epsilon\delta_{ij}. \tag{15}$$

T_{ij} is given by equation (14), while from (4)

$$\phi_{ij} = \phi_{ij,1} + \phi_{ij,2},$$

the two components being given by equations (5) and (6). Substituting and solving, we derive Rodi's result

$$m_{ij} = \frac{\langle u_i u_j \rangle}{k} - \tfrac{2}{3}\delta_{ij} \tag{16}$$

$$= \frac{(1-c_2)(P_{ij} - \frac{2}{3}P\delta_{ij})}{c_1\epsilon + (P-\epsilon)}. \tag{17}$$

Now $$P_{11} = 2P, \quad P_{22} = P_{33} = 0, \quad P_{12} = -u_2'^2\,\partial U_1/\partial x_2,$$

so that $$m_{11} = \frac{4}{3}\frac{(1-c_2)(P/\epsilon)}{c_1 + (P/\epsilon) - 1}, \tag{18}$$

$$m_{22} = m_{33} = -\tfrac{1}{2}m_{11}. \tag{19}$$

Also $$|m_{12}| = \frac{\langle -u_1 u_2 \rangle}{k} = (\tfrac{1}{2}m_{11} - \tfrac{3}{8}m_{11}^2)^{\frac{1}{2}}. \tag{20}$$

The constraint (19) is inevitable if the model of ϕ_{ij} does not permit anisotropy in the 23 plane. The relationship (20) is more striking, since it is independent of the model constants c_1, c_2 and of P/ϵ: it is, of course, related to the nature of the model. However the model does permit the individual m_{ij} to vary with P/ϵ: they are no longer fixed, as they were in the original formulations of Launder *et al.* and Gibson & Launder.

Table 3 compares the values of the anisotropy parameters m_{ij}, as measured in both experiments, with the values predicted by equation (17) for the appropriate value of P/ϵ. For the HGC experiment this is 1·55 (see equation (3)) while we have taken $P/\epsilon = 1$ for the CHC experiment, in which the advection is too small to measure. c_1 and c_2 have been put equal to 1·8 and 0·6 respectively, in accordance with Gibson & Launder's recommendations.

The prediction of m_{11} is good, and can be made almost perfect by raising c_2 to 0·625. (However, Launder *et al.* read the CHC value of m_{11} as 0·30, in agreement with the predicted value. Moreover, with this reading, the measured values of m_{11} and $|m_{12}|$ are compatible with equation (20).) This shows that the algebraic model can accommodate

	$\|m_{12}\|$	$\left\{\frac{3}{4}m_{11}\frac{u_2'^2}{k}\right\}^{\frac{1}{2}}$	
Experiment	Experimental	Experimental	Theoretical
CHC	0·34	0·32	0·34
HGC	0·30	0·32	0·36

TABLE 4. Correction of $|m_{12}|$ for anisotropy in the 23 plane.

both experiments, and that the difference between them is adequately accounted for by the differing values of P/ϵ.

The simplified model of $\phi_{ij,2}$ cannot predict anisotropy in the 23 plane. Experimentally, this is found to increase with increasing strain rate:

$$\frac{u_3'^2}{u_2'^2} = 1{\cdot}17 \quad \text{in CHC} \quad \left(\frac{\partial U_1}{\partial x_2} = 12{\cdot}9\,\text{s}^{-1}\right).$$

$$= 1{\cdot}50 \quad \text{in HGC} \quad \left(\frac{\partial U_1}{\partial x_2} = 48{\cdot}0\,\text{s}^{-1}\right).$$

It seems quite likely that a better model of $\phi_{ij,2}$, combined with algebraic modelling, could accommodate this variation.

The figures for $|m_{12}|$ are at first sight rather disconcerting, since the prediction is excellent at the low strain rate and poor at the high strain rate. However, it is probable that this anomaly is connected with the inherent failure to predict anisotropy in the 23 plane. Putting $ij = 12$ in equation (17), we find

$$|m_{12}| = \frac{(1-c_2)}{c_1\epsilon+P-\epsilon}u_2'^2\frac{\partial U_1}{\partial x_2} = \tfrac{4}{3}m_{11}\frac{u_2'^2}{\langle -u_1u_2\rangle}$$

from equation (18), or

$$|m_{12}| = \left\{\tfrac{3}{4}m_{11}\frac{u_2'^2}{k}\right\}^{\frac{1}{2}}. \tag{21}$$

Equation (20) is then derived by substituting for m_{22} from equation (19) and this introduces into the prediction of $|m_{12}|$ the incorrect value $u_3'^2/u_2'^2 = 1$. We can get round this difficulty by using experimental values for m_{11} and $u_2'^2/k$. (Since the predictions of m_{11} are good, it is immaterial whether we use experimental or theoretical values of this quantity.) The results of this exercise are shown in table 4. (The last column, which is taken from table 3, is included for comparison.) The anomaly has been more or less removed, and this makes it seem likely that a model of $\phi_{ij,2}$ less restricted than (6) would overcome the difficulty. (This better model would presumably alter the relationship (21). At present there is no way of knowing how this would affect $|m_{12}|$.)

HGC analyse their results by retaining the term $\phi_{ij,1}$ only. They find that the constant c_1 is direction dependent, and in particular that it is large and variable in the 3-direction. They also conclude (p. 677) 'that the linear inter-component energy transfer hypothesis is unlikely to be even a fair approximation'. The present work makes it seem likely that these conclusions follow from the ignoring of $\phi_{ij,2}$, and the predictions are much improved when this term is modelled after the manner of Naot *et al.* (1970), as interpreted by Gibson & Launder (1978) (see equation (6)). There is a clear need to account for the anisotropy in the 23 plane and the work of Launder *et al.* (1975) suggests (despite our criticisms) that this can be done within the framework of linear modelling. Only when this has been tried will it be possible to see whether the CHC and HGC experiments really do demand nonlinear modelling.

6. The magnitudes of the pressure–strain components

So far, attention has been concentrated on predicting the anisotropies, since these can be compared directly with measured quantities. It is also possible to compute the (non-zero) components of the pressure–strain tensor from the available experimental data, provided that the diffusion is ignored, and both CHC and HGC have done this. These components are of course derived rather than primary data, but they are instructive nonetheless. We shall now recalculate them from the HGC experiment only (the amendments on the errata sheet affect the values quoted in the paper) and will compare these with our recommended method of calculation.

Substituting measured values into the individual components of the balance equation (12), and identifying the transport with the advection term

$$\overline{U}_c \frac{d}{dx_1}\langle u_i u_j\rangle,$$

we find the experimental values to be (in units of $10^4\,\mathrm{cm^2\,s^{-3}}$)

$$\phi_{11} = -2P + \tfrac{2}{3}\epsilon \quad + \overline{U}_c \frac{d}{dx_1} u_1'^2$$
$$-11\cdot68 + 2\cdot51 + 1\cdot96 = -7\cdot21,$$
$$\phi_{22} = \tfrac{2}{3}\epsilon + \overline{U}_c \frac{d}{dx_2} u_2'^2$$
$$+2\cdot51 + 0\cdot88 = +3\cdot39,$$
$$\phi_{33} = \tfrac{2}{3}\epsilon + \overline{U}_c \frac{d}{dx_1} u_3'^2$$
$$+2\cdot51 + 1\cdot32 = +3\cdot83,$$
$$\phi_{12} = u_3'^2 \frac{dU_1}{dx_2} - \overline{U}_c \frac{d}{dx_1}\langle u_1 u_2\rangle$$
$$7\cdot83 - 0\cdot62 = +7\cdot21.$$

The values preducted by the recommended method of calculation using the *measured* anisotropies are listed below, with $c_1 = 1\cdot8$ and $c_2 = 0\cdot6$. The experimental values are in brackets, following the totals of the predicted values. The units are again $10^4\,\mathrm{cm^2\,s^{-3}}$, and the formulae are taken from equations (5) and (9):

$$\phi_{11} = -c_1\epsilon m_{11} - \tfrac{4}{3}c_2 P$$
$$-2\cdot27 - 4\cdot67 = -6\cdot94 \quad (\text{cf.} -7\cdot2),$$
$$\phi_{22} = -c_1\epsilon m_{22} + \tfrac{2}{3}c_2 P$$
$$+1\cdot81 + 2\cdot34 = +4\cdot15 \quad (\text{cf.} +3\cdot39),$$
$$\phi_{33} = -c_1\epsilon m_{33} + \tfrac{2}{3}c_2 P$$
$$+0\cdot46 + 2\cdot34 = +2\cdot80 \quad (\text{cf.} +3\cdot83),$$
$$\phi_{12} = +c_1\epsilon\,|m_{12}| + c_2 u_2'^2\, dU_1/dx_2$$
$$+2\cdot01 + 4\cdot70 = +6\cdot71 \quad (\text{cf.} +7\cdot21).$$

The predictions of ϕ_{11} and ϕ_{12} are good, the errors being 4 % and 8 % respectively. The sum $\phi_{22}+\phi_{33}$ is also well predicted, as it must be by continuity, but the division into the two components is not good. This is obviously due to the failure to deal adequately with the anisotropy of $\phi_{ij,2}$ in the 23 plane.

The magnitude of the $\phi_{ij,2}$ (mean field) component is always greater than that of the $\phi_{ij,1}$ (triple) term. It is, therefore, not surprising that HGC find it difficult to analyse the ϕ_{ij} in terms of $\phi_{ij,1}$ only.

7. Conclusions

The experiments of Harris *et al.* (1977) have been re-analysed using the representation of the pressure–strain correlation recommended by Gibson & Launder (1978), together with the algebraic modelling method of Rodi (1976). It is found that:

(1) Algebraic modelling is extremely good, although the flow is some way from local equilibrium ($P/\epsilon = 1{\cdot}55$) and the flow properties may therefore be predicted without integrating differential transport equations.

(2) Prediction of the longitudinal anisotropy, and of the longitudinal component of ϕ_{11}, is good.

(3) All other predictions are affected by the failure of the GL method to allow for the anisotropy of $\phi_{ij,2}$ in the 23 plane. There is substantial evidence that this failure is responsible for the major part of the errors of prediction.

(4) There is, therefore, a clear need to improve the Gibson & Launder model in this respect, perhaps along the lines suggested by Launder *et al.* (1975). However, it has been demonstrated that the postulate underlying the latter model for the mean field part of the pressure–strain correlation is too restricted.

(5) This analysis does not suggest any need to adopt a nonlinear model for the pressure–strain correlation.

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Appendix

(*a*) *Estimation of the dissipation from experimental data*

The value of the dissipation used in the main text ($\epsilon = 3{\cdot}76\times10^4\,\mathrm{cm^2\,s^{-3}}$; see equation (3)) is inferred from experimental measurements of the production and the advection. It can be inferred more directly from measured values of the longitudinal Taylor microscale λ_1 or the longitudinal integral scale L_1 (we use HGC's definitions of these quantities).

The variable ϵ is related to λ_1 by

$$\epsilon = \frac{20\nu k}{\lambda_1^2} = 2{\cdot}5\times10^4\,\mathrm{cm^2\,s^{-3}} \tag{A 1}$$

with $\lambda_1 = 0{\cdot}70\,\mathrm{cm}$ (from the errata sheet) and $\nu = 0{\cdot}15\,\mathrm{cm^2\,s^{-1}}$. This value, which agrees with that quoted in the paper, is 34 % too low. The discrepancy is not surprising, in view of the scatter in λ_1 shown in figure 4 of HGC.

The same figure shows that, as one would expect, L_1 can be measured much more accurately: however, to infer ϵ from this quantity, one must know something about the spectrum. Now L_1 is given by

$$L_1 = \frac{\pi}{2u_1'^2}\phi_1(0), \tag{A 2}$$

where $\phi_1(k_1)$ is the one-sided one-dimensional longitudinal spectrum. The simplest possible model for this function, which has the right form in the inertial range, is

$$\left.\begin{aligned}\phi_1(k_1) &= \tfrac{18}{55}Ko\,\epsilon^{\frac{2}{3}}k_0^{-\frac{5}{3}}, \quad k_1 < k_0,\\ &= \tfrac{18}{55}Ko\,\epsilon^{\frac{2}{3}}k_1^{-\frac{5}{3}}, \quad k_1 > k_0,\end{aligned}\right\} \tag{A 3}$$

k_0 being an adjustable parameter and Ko the Kolmogorov constant. This model gives

$$u_1'^2 = \int_0^\infty \phi_1(k_1)\,dk_1 = \tfrac{9}{11}Ko(\epsilon/k_0)^{\frac{2}{3}}$$

which we shall modify to

$$u_1'^2 = \tfrac{9}{11}\mu\,Ko(\epsilon/k_0)^{\frac{2}{3}}, \quad \mu < 1, \tag{A 4}$$

since the model (A 3) must overestimate the area under ϕ_1. It is then easy to show that

$$\epsilon = \frac{\pi}{5\mu^{\frac{3}{2}}}\left(\frac{11}{9Ko}\right)^{\frac{3}{2}}\frac{u_1'^3}{L_1} = 0{\cdot}508\frac{k^{\frac{3}{2}}}{L_1}\mu^{-\frac{3}{2}} \tag{A 5}$$

with $Ko = 1{\cdot}5$, while the ratio of $k^{\frac{1}{2}}$ to u_1' is taken from table 1. HGC give a similar formula, with a multiplying factor $(\frac{2}{3})^{\frac{3}{2}} = 0{\cdot}544$ (and $\mu = 1$). With $\mu = 1$ and $L_1 = 5{\cdot}3$ cm (from the errata sheet) equation (A 5) gives $\epsilon = 2{\cdot}5 \times 10^4\,\mathrm{cm^2\,s^{-3}}$ in precise agreement with equation (A 1), but this must be fortuitous.

HGC do not give spectra, and an attempt has therefore been made to estimate μ from figure 19 of CHC, which is a plot of $\psi = \phi_1(k_1)/\phi_1(0)$ as a function of $s = \eta k_1$, where $\eta = (\nu^3/\epsilon)^{\frac{1}{4}}$ is the Kolmogorov length scale (0·0346 cm in the CHC experiment). The experimental curve is quite well approximated by

$$\left.\begin{aligned}\psi(s) &= 1 && \text{for} \quad s < s_2 = 5{\cdot}5 \times 10^{-3},\\ &\propto s^{-1{\cdot}55} && \text{for} \quad s_2 < s < s_3 = 0{\cdot}2,\\ &\propto s^{-5{\cdot}5} && \text{for} \quad s_3 < s,\end{aligned}\right\} \tag{A 6}$$

giving

$$N = \int_0^\infty \psi\,ds = 0{\cdot}0163. \tag{A 7}$$

The curve is less well approximated by a model of the form (A 3) but, if the fitting is done at the knee at $s = s_3$, we find

$$N_{\text{model}} = 0{\cdot}0183,$$

implying

$$\mu_{\text{CHC}} = N/N_{\text{model}} = 0{\cdot}891.$$

If we assume that this value, derived from the CHC experiment, is also applicable to the HGC experiment (which does not seem unreasonable) then we find

$$\epsilon = 3{\cdot}3 \times 10^4\,\mathrm{cm^2\,s^{-3}},$$

which is only 11 % below the value (3) inferred from energy balance.

This seems satisfactory, but the calculation does not really hang together. It follows from equation (A 2) that

$$L_1 = \pi\eta/2N = 3{\cdot}3\ \text{cm},$$

N being taken from equation (A 7) (η is quoted above). A typical value for L_1 in the CHC experiment is 4·6 cm (see figure 17 of their paper) and it is not obvious how the discrepancy has arisen. This second method of estimating ϵ seems promising, but it needs more work.

(*b*) *Estimate of the diffusion term in the energy equation*

Daly & Harlow (1970) suggest that the contribution of diffusion to the transport term T_{ij} in equation (12) should be modelled by

$$D_{ij} = c_s' \frac{\partial}{\partial x_k}\left\{\frac{k}{\epsilon}\langle u_k u_l\rangle \frac{\partial}{\partial x_l}\langle u_i u_j\rangle\right\}.$$

The notation is that of Launder *et al.*, who recommend $c_s' = 0{\cdot}25$. With this model, the leading contribution to the diffusion term in the scalar energy equation is

$$c_s' \frac{\partial}{\partial x_1}\left(u_1'^2 \frac{k}{\epsilon}\frac{\partial k}{\partial x_1}\right) \simeq 2c_s' \frac{u_1'^2}{\epsilon}\left(\frac{\partial k}{\partial x_1}\right)^2$$

if we use algebraic modelling and ignore $d\epsilon/dx_1$. Using the values quoted in section 2, we find that this term is approximately

$$15\,\mathrm{cm^2\,s^{-3}}$$

while the other terms in equation (12) (or equation (1)) are all greater than $10^4\,\mathrm{cm^2\,s^{-3}}$. Unless the Daly–Harlow model is utterly wrong, diffusion is indeed negligible.

(*c*) *Disproof of the Launder* et al. *model for* $\phi_{ij,2}$

If the flow field is homogeneous, the mean field component $\phi_{ij,2}$ of the pressure–strain correlation is given by equation (10) where, in wavenumber space,

$$a_{lj}^{mi} = 2\int \frac{n_l n_j}{n^2} q_{mi}(\mathbf{n})\, d^3\mathbf{n}, \tag{A 8}$$

$q_{mi}(\mathbf{n})$ being the Fourier transform with respect to $\mathbf{r}$ of $\langle u_i(\mathbf{x})\, u_m(\mathbf{x}+\mathbf{r})\rangle$. ($\mathbf{n}$ is used to denote wavenumber since k has been pre-empted for the turbulent kinetic energy.) If the field is also isotropic, then

$$q_{mi}(\mathbf{n}) = \left(\delta_{in} - \frac{n_i n_m}{n^2}\right)\frac{E(n)}{4\pi n^2} \tag{A 9}$$

(see, e.g., Leslie 1973, chapter 2), where

$$\int_0^\infty E(n)\, dn = k \quad \text{(the turbulent kinetic energy).}$$

Substituting from (A 8) into (A 9), we find

$$a_{lj}^{mi} = (4\delta_{lj}\delta_{im} - \delta_{ij}\delta_{lm} - \delta_{il}\delta_{mj})\frac{2k}{15}, \tag{A 10}$$

which with equation (10) gives equation (6), with $c_2 = \frac{3}{5} = 0{\cdot}6$ as stated in the main text.

Launder *et al.* (1975) suggest that, when the field is not isotropic, a_{lj}^{mi} will be the most general linear functional of the stress tensor $\langle u_a u_b \rangle$. They show, taking account of the four kinematic constraints on these coefficients, that

$$a_{lj}^{mi} = \frac{10+4\gamma_2}{55}\delta_{lj}\langle u_m u_i\rangle - \frac{2+3\gamma_2}{11}(\delta_{ml}\langle u_i u_j\rangle + \delta_{mj}\langle u_i u_j\rangle + \delta_{il}\langle u_m u_j\rangle) + \gamma_2\delta_{mi}\langle u_l u_j\rangle + \left[-\frac{4+50\gamma_2}{55}\delta_{mi}\delta_{lj} + \frac{6+20\gamma_2}{55}(\delta_{ml}\delta_{ij}+\delta_{mj}\delta_{il})\right]k; \quad \text{(A 11)}$$

γ_2 is an adjustable parameter for which Launder *et al.* recommend the value 0·4. However, they recommend $c_1 = 1{\cdot}5$ while we have used Gibson & Launder's recommendation $c_1 = 1{\cdot}8$.

For a simple shear flow, equations (10) and (A 11) reduce to equations (11) of the main text. With $\gamma_2 = 0{\cdot}4$ and c_1 taken as either 1·5 or 1·8, the predictions of equation (11) are distinctly inferior to those of the simple model (6). It will now be shown that this is no accident and that the assumption underlying equation (A 11) is incorrect. Any disproof will suffice, and we shall give it for an axisymmetric (homogeneous) flow, since Herring (1974) has worked out the details of how such a flow should be represented.

He finds that the correlation tensor for such a flow may be written

$$q_{mi}(\mathbf{n}) = \sum_{\lambda=1}^{2}\sum_{s=0}^{\infty}\Phi_s^\lambda(n)\,P_s(\mu)\,e_m^\lambda(\mathbf{n})\,e_i^\lambda(\mathbf{n}). \quad \text{(A 12)}$$

Here

$$\mu = \cos(\mathbf{a},\mathbf{n}), \quad \text{(A 13)}$$

a being the unit vector along the symmetry axis; $e_m^1(\mathbf{n})$ and $e_m^2(\mathbf{n})$ are the eigenvectors of $q_{mi}(\mathbf{n})$, $P_s(\mu)$ is the Legendre polynomial of order s; the $\Phi_s^\lambda(n)$, which depend on scalar n only, are the eigenvalues of the correlation function. There must, of course, be three eigenvectors and three eigenvalues. The third eigenvector is the unit vector parallel to **n**. In a representation in which the 1-direction is along **a**,

$$e^3(\mathbf{n}) = (\mu, (1-\mu^2)^{\frac{1}{2}}\cos\phi, (1-\mu^2)^{\frac{1}{2}}\sin\phi), \quad \text{(A 14)}$$

ϕ being an angle variable in the 23 plane perpendicular to **a**: by continuity, the corresponding eigenvalue is zero.

Herring takes e^1 and e^2 to be in the directions $\mathbf{a}\wedge\mathbf{n}$ and $\mathbf{a}\wedge(\mathbf{a}\wedge\mathbf{n})$: in the representation

$$e^1(\mathbf{n}) = (0, \sin\phi, -\cos\phi), \quad \text{(A 15)}$$

$$e^2(\mathbf{n}) = (-(1-\mu^2)^{\frac{1}{2}}, \mu\cos\phi, \mu\sin\phi). \quad \text{(A 16)}$$

Since the angular dependence of the tensor (A 12) is on μ only, it must be axisymmetric: in a more general tensor, the Legendre polynomials will be replaced by zonal harmonics.

From (A 14), (A 15) and (A 16)

$$\sum_{\lambda=1}^{3} e_m^\lambda e_i^\lambda = \delta_{im}$$

and since $e_m^3 = n_m/n$ by definition, it follows that

$$\sum_{\lambda=1}^{2} e_m^\lambda e_i^\lambda = P_{mi}(\mathbf{n})$$

(see Leslie 1973). Thus the usual isotropic tensor is generated by

$$\Phi_0^1(n) = \Phi_0^2(n) = q(n); \quad \Phi_s^\lambda(n) = 0 \quad \text{for} \quad s \geqslant 1.$$

The simplest anisotropic tensor is generated by putting

$$\Phi_0^1 \neq 0, \quad \text{rest zero,} \tag{A 17}$$

and this gives

$$\langle u_m u_i \rangle = \int q_{mi}(n)\, d^3n = \tfrac{1}{2}(\delta_{m2}\delta_{i2} + \delta_{m3}\delta_{i3})\, F_0^1 \quad \text{where } F_0^1, = \int_0^\infty 4\pi n^2 \Phi_0^1(n)\, dn.$$

Thus $\langle u_2^2 \rangle = \langle u_3^2 \rangle = k$ (kinetic energy) $= \frac{1}{2}F_0^1$, rest zero. (A 18)

From (A 17), (A 12) and (A 8)

$$\begin{aligned} a_{lj}^{mi} &= 2\int \frac{n_l n_j}{n^2}\, \Phi_0^1(n)\, e_m^1(\mathbf{n})\, e_i^1(\mathbf{n})\, d^3\mathbf{n} \\ &= 4k \times \frac{1}{2}\int_{-1}^{+1} d\mu \,.\, \frac{1}{2\pi}\oint d\phi \frac{n_l n_j}{n^2}\, e_m^1(\mathbf{n})\, e_j^1(\mathbf{n}). \end{aligned} \tag{A 19}$$

For example

$$a_{11}^{22} = 4k \times \frac{1}{2}\int_{-1}^{+1} d\mu \frac{1}{2\pi}\oint d\phi\, \mu^2 \sin^2\phi = \tfrac{2}{3}k \tag{A 20}$$

and similarly

$$a_{22}^{22} = \tfrac{1}{3}k, \quad a_{33}^{22} = k. \tag{A 21}$$

Now, when we substitute the tensorial form (A 18) into the Launder *et al.* constitutive relation, we find

$$\left.\begin{aligned} a_{11}^{22} &\overset{?}{=} \frac{46 - 30\gamma_2}{55}k, \\ a_{22}^{22} &\overset{?}{=} \frac{18 + 5\gamma_2}{55}k, \\ a_{33}^{22} &\overset{?}{=} \frac{46 + 25\gamma_2}{55}k. \end{aligned}\right\} \tag{A 22}$$

There is no value of γ_2 which will bring all three components of (A 22) into agreement with (A 20) and (A 21) (whence the ?s in (A 22)), and this disproves Launder *et al.*'s conjecture.

The cause of the difficulty is clear. The tensor $\langle u_m u_i \rangle$ is determined by the zero-order angular components $P_0(\mu)$ only, whereas higher angular harmonics enter into the coefficients a_{lj}^{mi}. This point is made very clearly if we replace (A 17) by

$$\Phi_0^1(n),\ \Phi_2^1(n) \neq 0, \quad \text{rest zero.}$$

This change leaves (A 18) and therefore (A 22) unaffected, but it does alter the a_{lj}^{mi}.

A referee has pointed out that Lumley (1978) has disproved the Launder *et al.* model for $\phi_{ij,2}$, using invariance arguments. His reasoning must be related to that given above, but the connexion is not immediately obvious.

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